

# MAXIMUM PRINCIPLE FOR LINEAR-CONVEX BOUNDARY CONTROL PROBLEMS APPLIED TO OPTIMAL INVESTMENT WITH VINTAGE CAPITAL

SILVIA FAGGIAN<sup>1</sup>

**ABSTRACT.** The paper concerns the study of the Pontryagin Maximum Principle for an infinite dimensional and infinite horizon boundary control problem for linear partial differential equations. The optimal control model has already been studied both in finite and infinite horizon with Dynamic Programming methods in a series of papers by the same author et al. [26, 27, 28, 29, 30]. Necessary and sufficient optimality conditions for open loop controls are established. Moreover the co-state variable is shown to coincide with the spatial gradient of the value function evaluated along the trajectory of the system, creating a parallel between Maximum Principle and Dynamic Programming. The abstract model applies, as recalled in one of the first sections, to optimal investment with vintage capital.

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**Keywords:** Linear convex control, Boundary control, Hamilton–Jacobi–Bellman equations, Optimal investment problems, Vintage capital.

## 1. INTRODUCTION

The paper concerns the study of the Pontryagin Maximum Principle for a infinite dimensional and infinite horizon boundary control problem for linear partial differential equations. More precisely, we take into account a state equation of type

$$(1.1) \quad \begin{cases} y'(\tau) = A_0 y(\tau) + B u(\tau), & \tau \in [t, +\infty) \\ y(t) = x \in H, \end{cases}$$

where  $H$  is the state space,  $y : [t, +\infty) \rightarrow H$  is the trajectory,  $U$  is the control space and  $u : [t, +\infty) \rightarrow U$  is the control,  $A_0 : D(A_0) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup of linear operators  $\{e^{\tau A_0}\}_{\tau \geq 0}$  on  $H$ , and the control operator  $B$  is linear and *unbounded*, say  $B : U \rightarrow [D(A_0^*)]'$ . Besides, we consider a cost functional given by

$$(1.2) \quad J_\infty(t, x, u) = \int_t^{+\infty} e^{-\lambda \tau} [g_0(y(\tau)) + h_0(u(\tau))] d\tau$$

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<sup>1</sup>LUM “Jean Monnet”, Casamassima (Bari), I-70010.

where the functions  $\varphi_0$  and, for all fixed  $\tau$  in  $[0, T]$ ,  $g_0$  and  $h_0$  are convex functions as better specified later. The state equation (1.1) is then associated to a costate or dual system, given by

$$(1.3) \quad \pi'(\tau) = (\lambda - A_0^*)\pi(\tau) - g'_0(y(\tau)), \quad \tau \in [t, +\infty)$$

where  $\pi : [t, +\infty) \rightarrow H$  (the dual variable, or co-state of the system) is the unknown, and  $y = y(\cdot; t, x, u)$  is the trajectory starting at  $x$  at time  $t$  and driven by control  $u$ , given by (1.1). A transversality condition

$$(1.4) \quad \lim_{T \rightarrow +\infty} \pi(T) = 0,$$

is also assumed.

We recall that the problem of minimizing  $J_\infty(t, x, u)$  with respect to  $u$  over the Banach space

$$L_\lambda^p(t, +\infty; U) = \{u : [t, +\infty) \rightarrow U : \tau \mapsto u(\tau)e^{-\frac{\lambda}{p}\tau} \in L^p(t, +\infty; U)\}, \quad p \geq 2,$$

was studied in a previous paper by Faggian and Gozzi [30], deriving existence and uniqueness for the associated Hamilton-Jacobi-Bellman (briefly, HJB) equation

$$(1.5) \quad -\lambda\psi(x) + \langle \psi'(x), A_0x \rangle - h_0^*(-B^*\psi'(x)) + g(x) = 0,$$

and a feedback formula for optimal controls in terms of the spatial gradient of the value function. All such results are recalled in Section 2.

Thus in the present paper some further results are achieved:

- we establish necessary and sufficient optimality conditions for open loop controls, that is, the Pontryagin Maximum Principle (Theorem 4.7) ;
- we show that the co-state variable coincides with the spatial gradient of the value function evaluated along the trajectory of the state equation, creating a connection between Maximum Principle and Dynamic Programming (Lemma 4.8).

Moreover, in Section 3 the reader will find the economic application to optimal investment with vintage capital.

**1.1. Bibliographical notes.** It is well known that control problems with unbounded control operator  $B$  arise when we rephrase into abstract terms some boundary control problem for PDEs (or, more generally, problems with control on a subdomain). Indeed, we motivate our framework with the application to the economic problem of optimal investment with vintage capital in the framework by Barucci and Gozzi [12] [13] that we describe in detail in Section 3. Similar problems with unbounded control operator have been discussed in a series of papers by this author and others. The unconstrained case has been studied both in the case of finite and infinite horizon [26, 27, 30] while [29] contains the finite horizon case with constrained controls. The (finite horizon) case with both boundary control and state constraints is treated in [28].

Some further references on *boundary control* in infinite dimension follow. We recall that such problems have been studied in the framework of classical/strong solutions and in that

of viscosity solutions. Regarding Dynamic Programming in the classical/strong framework, the available results mainly regard the case of linear systems and quadratic costs (where HJB reduces to the operator Riccati equation). The reader is then referred *e.g.* to the book by Lasiecka and Triggiani [41], to the book by Bensoussan, Da Prato, Delfour and Mitter [14], and, for the case of nonautonomous systems, to the papers by Acquistapace, Flandoli and Terreni [2, 3, 4, 5]. For the case of a linear system and a general convex cost, we mention the papers by this author [24, 25, 26, 27]. On Pontryagin maximum principle for boundary control problems we mention again the book by Barbu and Precupanu (Chapter 4 in [11]).

For viscosity solutions and HJB equations in infinite dimension we mention the series of papers by Crandall and Lions [18] where also some boundary control problem arises. Moreover, for boundary control we mention Gozzi, Cannarsa and Soner [17] and the paper by Cannarsa and Tessitore [19] on existence and uniqueness of viscosity solutions of HJB. We note also that a verification theorem in the case of viscosity solutions has been proved in some finite dimensional case in the book by Yong and Zhou [43]. We finally mention the paper by Fabbri [23] where the author derives an existence and uniqueness result for the viscosity solution of HJB associated to optimal investment with vintage capital (with infinite horizon and without constraints), that is the application of Section 3 of the present paper, obtaining the results by making use of the specific properties of the state equation, while no result is there provided for the general problem.

We mention also some fundamental papers and books on the case of *distributed control* in the classical/strong framework such as the works by Barbu and Da Prato [7, 8, 9] for some linear convex problems, to Di Blasio [20, 21] for the case of constrained control, to Cannarsa and Di Blasio [16] for the case of state constraints, to Barbu, Da Prato and Popa [10] and to Gozzi [35, 36, 37] for semilinear systems.

Regarding applications, on control on a subdomain (boundary or point control) we refer the reader to the many examples contained in the books by Lasiecka and Triggiani [41], and by Bensoussan *et al* [14]. Moreover, for economic models with vintage capital the reader may see the papers by Barucci and Gozzi [12], [13], the papers by Feichtinger, Hartl, Kort, Veliov et al. [31, 32, 33, 34], and for population dynamic the book by Iannelli [40], the paper by Anița, Iannelli, Kim and Park [1], and the papers by Almeder, Caulkins, Feichtinger, Tragler, and Veliov [6] and references therein.

## 2. PRELIMINARIES

We here recall all the relevant results that are needed in the sequel. According to the notation in [30], if  $X$  and  $Y$  are Banach spaces, we denote by  $|\cdot|_X$  the norm on  $X$ , by  $|\cdot|$

the euclidean norm in  $\mathbb{R}$ , and we set

$$\begin{aligned} Lip(X; Y) &= \{f : X \rightarrow Y : [f]_L := \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|_Y}{|x - y|_X} < +\infty\} \\ C_{Lip}^1(X) &:= \{f \in C^1(X) : [f']_L < +\infty\} \\ \mathcal{B}_r(X, Y) &:= \{f : X \rightarrow Y : |f|_{\mathcal{B}_r} := \sup_{x \in X} \frac{|f(x)|_Y}{1 + |x|_X^r} < +\infty\}, \quad \mathcal{B}_r(X) := \mathcal{B}_r(X, \mathbb{R}). \end{aligned}$$

Moreover we set

$$\Sigma_0(X) := \{w \in \mathcal{B}_2(X) : w \text{ is convex, } w \in C_{Lip}^1(X)\}.$$

All the spatial derivatives above have to be intended as Frechét differentials.

Then we consider two Hilbert spaces  $V, V'$ , being dual spaces, which we do not identify for reasons which are recalled in Remark 2.2 and we denote the duality pairing by  $\langle \cdot, \cdot \rangle$ . We set  $V'$  as the state space of the problem, and denote with  $U$  the control space, being  $U$  another Hilbert space. The state space is  $V'$  and the control space is  $U$ . For any fixed  $x$  in  $V'$  and  $t > 0$  and  $\tau \geq t$ , the solution to the state equation in  $V'$  is given by variation of constant formula by

$$(2.1) \quad y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}Bu(\sigma)d\sigma, \quad \tau \in [t, +\infty[,$$

while the target functional is of type

$$(2.2) \quad J_\infty(t, x, u) := \int_t^{+\infty} e^{-\lambda\tau} [g_0(y(\tau)) + h_0(u(\tau))] d\tau.$$

We assume the following hypotheses hold:

**Assumptions 2.1.** 1.  $A : D(A) \subset V' \rightarrow V'$  is the infinitesimal generator of a strongly continuous semigroup  $\{e^{\tau A}\}_{\tau \geq 0}$  on  $V'$ ;

2.  $B \in L(U, V')$ ;

3. there exists  $\omega \geq 0$  such that  $|e^{\tau A}x|_{V'} \leq e^{\omega\tau}|x|_{V'}, \forall \tau \geq 0$ ;

4.  $g_0, \phi_0 \in \Sigma_0(V')$

5.  $h_0$  is convex, lower semi-continuous,  $\partial_u h_0$  is injective.

6.  $h_0^*(0) = 0, h_0^* \in \Sigma_0(V)$ ;

7.  $\exists a > 0, \exists b \in \mathbb{R}, \exists p \geq 2 : h_0(u) \geq a|u|_U^p + b, \forall u \in U$ ;

Moreover, either

8.a  $p > 2, \lambda > 2\omega$ .

or

8.b  $\lambda > \omega$ , and  $g_0, \phi_0 \in \mathcal{B}_1(V')$ .

**Remark 2.2.** We do not identify  $V$  and  $V'$  for in the applications the problem is naturally set in a Hilbert space  $H$ , such that  $V \subset H \equiv H' \subset V'$  (with all bounded inclusions). Indeed, in order to avoid the discontinuities due to the presence of  $B$ , as they appear in (1.1)(1.2), we work in the extended state space  $V'$  related to  $H$  in the following way:  $V$  is

the Hilbert space  $D(A_0^*)$  endowed with the scalar product  $(v|w)_V := (v|w)_H + (A_0^*v|A_0^*w)_H$ ,  $V'$  is the dual space of  $V$  endowed with the operator norm. Then assume that  $B \in L(U, V')$ , and extend the semigroup  $\{e^{tA_0}\}_{t \geq 0}$  on  $H$  to a semigroup  $\{e^{tA}\}_{t \geq 0}$  on the space  $V'$ , having infinitesimal generator  $A$ , a proper extension of  $A_0$ . The reader is referred to [27] for a detailed treatment. The coefficient  $\omega$  could be any real number, but is assumed positive in order to avoid double proofs for positive and negative signs.  $\square$

**Remark 2.3.** Note that the functions  $g$  and  $\varphi$  arising from applications usually appear to be defined and  $C^1$  on  $H$ , not on the larger space  $V'$ . Then, we here need to *assume* that they can be extended to  $C^1$ -regular functions on  $V'$  - which is a non trivial issue. We refer the reader to [26], [27] and [30] for a thorough discussion on this issue.  $\square$

The functional  $J_\infty(t; x, u)$  has to be minimized with respect to  $u$  over the set of admissible controls

$$(2.3) \quad L_\lambda^p(t, +\infty; U) = \{u : [t, +\infty) \rightarrow U ; \tau \mapsto u(\tau)e^{-\frac{\lambda\tau}{p}} \in L^p(t, +\infty; U)\},$$

which is Banach space with the norm

$$\|u\|_{L_\lambda^p(t, +\infty; U)} = \int_t^{+\infty} |u(\tau)|_U^p e^{-\lambda\tau} d\tau = \|e^{-\frac{\lambda(\cdot)}{p}} u\|_{L^p(t, +\infty; U)}.$$

The value function is then defined as

$$Z_\infty(t, x) = \inf_{u \in L_\lambda^p(t, +\infty; U)} J_\infty(t, x, u).$$

As it is easy to check that

$$Z_\infty(t, x) = e^{-\lambda t} Z_\infty(0, x)$$

one may associate to the problem the following stationary HJB equation

$$(2.4) \quad -\lambda\psi(x) + \langle \psi'(x), Ax \rangle - h_0^*(-B^*\psi'(x)) + g(x) = 0,$$

whose candidate solution is the function  $Z_\infty(0, \cdot)$ .

We will use the following definition of solution for equation (2.4).

**Definition 2.4.** A function  $\psi$  is a classical solution of the stationary HJB equation (2.4) if it belongs to  $\Sigma_0(V')$  and satisfies (2.4) for every  $x \in D(A)$ .

**Theorem 2.5.** Let Assumptions 2.1 hold. Then there exists a unique classical solution  $\Psi$  to (2.4) and it is given by the value function of the optimal control problem, that is

$$\Psi(x) = Z_\infty(0, x) = \inf_{u \in L_\lambda^p(0, +\infty; U)} J_\infty(0, x, u).$$

Once we have established that  $\Psi$  is the classical solution to the stationary HJB equation, and that it is differentiable, we can build optimal feedbacks and prove the following theorem.

**Theorem 2.6.** *Let Assumptions 2.1 hold. Let  $t \geq 0$  and  $x \in V'$  be fixed. Then there exists a unique optimal pair  $(u^*, y^*)$ . The optimal state  $y^*$  is the unique solution of the Closed Loop Equation*

$$(2.5) \quad y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}B(h_0^*)'(-B^*\Psi'(y(s)))d\sigma, \quad \tau \in [t, +\infty[.$$

while the optimal control  $u^*$  is given by the feedback formula

$$u^*(s) = (h_0^*)'(-B^*\Psi'(y^*(s))).$$

### 3. THE MOTIVATING EXAMPLE

We here describe our motivating example: the infinite horizon problem of optimal investment with vintage capital, in the setting introduced by Barucci and Gozzi [12][13], and later reprised and generalized by Feichtinger et al. [31, 32, 33], and by Faggian [26, 27].

The capital accumulation is described by the following system

$$(3.1) \quad \begin{cases} \frac{\partial y(\tau, s)}{\partial \tau} + \frac{\partial y(\tau, s)}{\partial s} + \mu y(\tau, s) = u_1(\tau, s), & (\tau, s) \in ]t, +\infty[ \times ]0, \bar{s}] \\ y(\tau, 0) = u_0(\tau), & \tau \in ]t, +\infty[ \\ y(t, s) = x(s), & s \in [0, \bar{s}] \end{cases}$$

with  $t > 0$  the initial time,  $\bar{s} \in [0, +\infty]$  the maximal allowed age, and  $\tau \in [0, T[$  with horizon  $T = +\infty$ . The unknown  $y(\tau, s)$  represents the amount of capital goods of age  $s$  accumulated at time  $\tau$ , the initial datum is a function  $x \in L^2(0, \bar{s})$ ,  $\mu > 0$  is a depreciation factor. Moreover,  $u_0 : [t, +\infty[ \rightarrow \mathbb{R}$  is the investment in new capital goods ( $u_0$  is the boundary control) while  $u_1 : [t, +\infty[ \times [0, \bar{s}] \rightarrow \mathbb{R}$  is the investment at time  $\tau$  in capital goods of age  $s$  (hence, the distributed control). Investments are jointly referred to as the control  $u = (u_0, u_1)$ .

Besides, we consider the firm profits represented by the functional

$$I(t, x; u_0, u_1) = \int_t^{+\infty} e^{-\lambda\tau} [R(Q(\tau)) - c(u(\tau))] d\tau$$

where, for some given measurable coefficient  $\alpha$ , we have that

$$Q(\tau) = \int_0^{\bar{s}} \alpha(s) y(\tau, s) ds$$

is the output rate (linear in  $y(\tau)$ )  $R$  is a concave revenue from  $Q(\tau)$  (i.e., from  $y(\tau)$ ). Moreover we have

$$c(u_0(\tau), u_1(\tau)) = \int_0^{\bar{s}} c_1(s, u_1(\tau, s)) ds + c_0(u_0(\tau)),$$

with  $c_1$  indicating the investment cost rate for technologies of age  $s$ ,  $c_0$  the investment cost in new technologies, including adjustment-innovation,  $c_0, c_1$  convex in the control variables.

The entrepreneur's problem is that of maximizing  $I(t, x; u_0, u_1)$  over all state–control pairs  $\{y, (u_0, u_1)\}$  which are solutions in a suitable sense of equation (3.1). Such problems are known as *vintage capital* problems, for the capital goods depend jointly on time  $\tau$  and on age  $s$ , which is equivalent to their dependence from time and vintage  $\tau - s$ .

When rephrased in an infinite dimensional setting, with  $H := L^2(0, \bar{s})$  as state space, the state equation (3.1) can be reformulated as a linear control system with an unbounded control operator, that is

$$(3.2) \quad \begin{cases} y'(\tau) = A_0 y(\tau) + Bu(\tau), & \tau \in ]t, +\infty[; \\ y(t) = x, \end{cases}$$

where  $y : [t, +\infty[ \rightarrow H$ ,  $x \in H$ ,  $A_0 : D(A_0) \subset H \rightarrow H$  is the infinitesimal generator of a strongly continuous semigroup  $\{e^{A_0 t}\}_{t \geq 0}$  on  $H$  with domain  $D(A_0) = \{f \in H^1(0, \bar{s}) : f(0) = 0\}$  and defined as  $A_0 f(s) = -f'(s) - \mu f(s)$ , the control space is  $U = \mathbb{R} \times H$ , the control function is a couple  $u \equiv (u_0, u_1) : [t, +\infty[ \rightarrow \mathbb{R} \times H$ , and the control operator is given by  $Bu \equiv B(u_0, u_1) = u_1 + u_0 \delta_0$ , for all  $(u_0, u_1) \in \mathbb{R} \times H$ ,  $\delta_0$  being the Dirac delta at the point 0. Note that, although  $B \notin L(U, H)$ , is  $B \in L(U, D(A_0^*)')$ . The reader can find in [12] the (simple) proof of the following theorem, which we will exploit in a short while.

**Theorem 3.1.** *Given any initial datum  $x \in H$  and control  $u \in L_\lambda^p(t, +\infty; U)$  the mild solution of the equation (3.2)*

$$y(s) = e^{(s-t)A} x + \int_t^s e^{(s-\tau)A} Bu(\tau) d\tau$$

*belongs to  $C([t, +\infty); H)$ .*

Following Remark 2.2, we then set

$$V = D(A_0^*) = \{f \in H^1(0, \bar{s}) : f(\bar{s}) = 0\}$$

and  $V' = D(A_0^*)'$ . Regarding the target functional, we set

$$J_\infty(t, x; u) := -I(t, x; u_0, u_1),$$

with:

$$\begin{aligned} g_0 : V' &\rightarrow \mathbb{R}, \quad g_0(x) = -R(\langle \alpha, x \rangle), \\ h_0 : U &\rightarrow \mathbb{R}, \quad h_0(u) = c_0(u_0) + \int_0^{\bar{s}} c_1(s, u_1(s)) ds. \end{aligned}$$

**Remark 3.2.** Here the extension of the datum  $g_0$  to  $V'$  is straightforward, as long as we assume that  $\alpha \in V$  and replace scalar product in  $H$  with the duality in  $V, V'$ .

Note further that  $\omega = 0$ ,  $\lambda > 0$  (the type of the semigroup is negative and equal to  $-\mu$ ).

□

As the problem now fits into our abstract setting, the main results of the previous sections apply to the economic problem when data  $R, c_0, c_1$  satisfy Assumption 2.1[8.a] or [8.b]. In particular, such thing happens in the following two interesting cases:

- If we assume, for instance, that  $R$  is a concave,  $C^1$ , sublinear function (for example one could take  $R$  quadratic in a bounded set and then take its linear continuation, see e.g. [31, 33]), and  $c_0, c_1$  quadratic functions of the control variable, then Assumption 2.1[8.b] holds.
- Assumption 2.1[8.a] is instead satisfied when  $R$  is, for instance, quadratic - as it occurs in some other meaningful economic problems - and  $c_0, c_1$  are equal to  $+\infty$  outside some compact interval, and equal to any convex *l.s.c.* function otherwise. Such case corresponds to that of constrained controls (controls that violate the constrain yield infinite costs).

In these cases, Theorems 2.5, 2.6 hold true. In particular Theorem 2.6 states the existence of a unique optimal pair  $(u^*, y^*)$  for any initial datum  $x \in V'$ . Note that in general the optimal trajectory  $y^*$  lives in  $V'$ . However, since the economic problem makes sense in  $H$ , we would now like to infer that whenever  $x$  (the initial age distribution of capital) lies in  $H$ , then the whole optimal trajectory lives in  $H$ . Indeed, this is guaranteed by Theorem 3.1.

All these results allow to perform the analysis of the behavior of the optimal pairs and to study phenomena such as the diffusion of new technologies (see e.g. [12, 13]) and the anticipation effects (see e.g. [31, 33]). With respect to the results in [12, 13], here also the case of nonlinear  $R$  (which is particularly interesting from the economic point of view, as it takes into account the case of large investors) is considered. With respect to the results in [31, 33], here the existence of optimal feedbacks yields a tool to study more deeply the long run behavior of the trajectories, like the presence of long run equilibrium points and their properties.

#### 4. THE MAXIMUM PRINCIPLE

Let  $x \in V'$  and  $t \geq 0$  be fixed, and consider the dual system associated to (1.1), that is

$$(4.1) \quad \pi'(\tau) = (\lambda - A_0^*)\pi(\tau) - g'_0(y(\tau)), \quad \tau \in [t, +\infty)$$

where  $\pi : [t, +\infty) \rightarrow V$  (the dual variable, or co-state of the system) is the unknown, and  $y = y(\cdot; t, x, u)$  is the trajectory starting at  $x$  at time  $t$  and driven by control  $u$ , given by (2.1). We assume such equation is also subject to the following transversality condition

$$(4.2) \quad \lim_{T \rightarrow +\infty} \pi(T) = 0.$$

We denote any solution of (4.1)(4.2) also by  $\pi(\cdot; t, x, u)$  or by  $\pi(\cdot; t, x)$  to remark its dependence on the data.

**Definition 4.1.** *Let Assumptions 2.1 [1-7] be satisfied. We define the mild solution of (4.1)(4.2) as the function  $\pi : [t, +\infty) \rightarrow V$  given by*

$$(4.3) \quad \pi(\tau) = \int_{\tau}^{+\infty} e^{(A_0^* - \lambda)(\sigma - \tau)} g'_0(y(\sigma)) d\sigma.$$

In the sequel we show that such definition is natural.



**Lemma 4.2.** *Let Assumption 2.1 [1-7] be satisfied, and assume  $p \geq 2$  and  $\lambda > 2\omega$ . Then  $\pi$  given by (4.3) is well defined and belongs to  $C^0(t, +\infty; V)$ .*

Moreover:

(i) if  $p > 2$  then  $\pi \in L_\lambda^q(t, +\infty; V)$ ;

(ii) if  $p = 2$  then  $\pi \in L_{\lambda+\varepsilon}^2(t, +\infty; V) \cap L^2(t, T; V)$ ,  $\forall T < +\infty$ ,  $\varepsilon > 0$ .

*Proof.* Note that by assumptions on  $g'_0$ , the integrand in (4.3) can be estimated as follows

$$|e^{(A_0^* - \lambda)(\sigma - \tau)} g'_0(y(\sigma))|_V \leq |g'_0|_{B_1} e^{-(\lambda - \omega)(\sigma - \tau)} (1 + |y(\sigma)|_{V'}),$$

hence to prove the first assertion it is enough to show that  $\sigma \mapsto e^{-(\lambda - \omega)\sigma} |y(\sigma)|_{V'}$  is in  $L^1(\tau, +\infty)$ .

By Lemma 4.5 contained in [30] one has

$$(4.4) \quad \begin{aligned} |y(\sigma)|_{V'} &\leq C e^{\omega\sigma} \left( |x|_{V'} + \int_t^\sigma e^{-\omega r} |u(r)|_U dr \right) \\ &\leq C_1 e^{\omega\sigma} (1 + \rho(t, \sigma)^{\frac{1}{q}}) \end{aligned}$$

for suitable constants  $C$  (depending only on  $t$ ) and  $C_1$  (depending from  $t$ ,  $x$ , and  $u$ ), where

$$\rho(t, \sigma) = \begin{cases} |e^{q(\frac{\lambda}{p} - \omega)t} - e^{q(\frac{\lambda}{p} - \omega)\sigma}| & \lambda \neq \omega p \\ |t - \sigma| & \lambda = \omega p. \end{cases}$$

Hence

$$e^{-(\lambda - \omega)\sigma} |y(\sigma)|_{V'} \leq C_1 e^{-(\lambda - 2\omega)\sigma} (1 + \rho(t, \sigma)^{\frac{1}{q}}),$$

so that in the case  $\lambda \neq \omega p$  one obtains

$$(4.5) \quad e^{-(\lambda - 2\omega)\sigma} \rho(t, \sigma)^{\frac{1}{q}} \leq |e^{-(\lambda - 2\omega)\sigma} - e^{-(\lambda - q\omega)\sigma}|^{\frac{1}{q}} \leq C_2 e^{-\frac{1}{q}[\lambda - 2\omega]\sigma}$$

for a suitable constant  $C_2$ , whereas in the case  $\lambda = \omega p$  one has

$$(4.6) \quad e^{-(\lambda - 2\omega)\sigma} \rho(t, \sigma)^{\frac{1}{q}} \leq e^{-(\lambda - 2\omega)\sigma} |t - \sigma|^{\frac{1}{q}}.$$

Similar estimates lead to the proof that  $\pi \in C^0(t, +\infty; V)$ .

Next we prove that if  $p > 2$  then  $\pi \in L_\lambda^q(t, +\infty; V)$ . From the estimates above, one has

$$(4.7) \quad \begin{aligned} |\pi(\tau)|_V &\leq \frac{|g'_0|_{B_1}}{\lambda - \omega} + \frac{|g'_0|_{B_1} C e^{\omega\tau}}{\lambda - 2\omega} + |g'_0|_{B_1} C e^{(\lambda - \omega)\tau} \int_\tau^{+\infty} e^{-(\lambda - 2\omega)\sigma} \rho(t, \sigma)^{\frac{1}{q}} d\sigma \\ &\equiv \gamma_1(\tau) + \gamma_2(\tau) + \gamma_3(\tau). \end{aligned}$$

The functions  $\gamma_1$  and  $\gamma_2$  are trivially in  $L_\lambda^q(t, +\infty; \mathbb{R})$ . Regarding  $\gamma_3$ , in the case  $\lambda = \omega p$ , let  $\delta > 0$  such that  $\lambda > \max\{2\omega, q(\omega + \delta)\}$ , so that there exists some  $T > \tau$  such that

$$\sigma \geq T \Rightarrow e^{-\delta\sigma} |t - \sigma|^{\frac{1}{q}} \leq 1.$$

Then

$$\begin{aligned}
\int_{\tau}^{+\infty} e^{-(\lambda-2\omega)\sigma} |t-\sigma|^{\frac{1}{q}} d\sigma &\leq \int_{\tau}^T e^{-(\lambda-2\omega)\sigma} |T-t|^{\frac{1}{q}} d\sigma + \int_T^{+\infty} e^{-(\lambda-2\omega-\delta)\sigma} d\sigma \\
&\leq \frac{e^{-(\lambda-2\omega)\tau}}{\lambda-2\omega} |T-t|^{\frac{1}{q}} + \frac{e^{-(\lambda-2\omega-\delta)\tau}}{\lambda-2\omega-\delta} \\
&\equiv \gamma_4(\tau) + \gamma_5(\tau)
\end{aligned}$$

Now for a suitable constant  $C_3$  one has

$$(4.8) \quad e^{-\lambda\tau} e^{q(\lambda-\sigma)\tau} [\gamma_4(\tau)^q \vee \gamma_5(\tau)^q] \leq C_3 e^{-[\lambda-(\omega+\delta)q]\tau}$$

which is integrable functions in  $[t, +\infty)$ . In the case  $\lambda \neq \omega p$ , by means of (4.5) one has

$$\begin{aligned}
(4.9) \quad e^{-\lambda\tau} \gamma_3(\tau)^q &\leq \left[ \frac{|g'_0|_{B_1} C C_1 q}{\lambda-2\omega} \right]^q e^{-\lambda\tau} e^{q(\lambda-\omega)\tau} e^{-[\lambda-2\omega]\tau} \\
&\leq C_4 e^{-(2-q)(\lambda-\omega)\tau}
\end{aligned}$$

for a suitable constant  $C_4$ , which implies  $\gamma_3 \in L_{\lambda}^q(t, +\infty; \mathbb{R})$  also in this case.

Finally, note that if  $p = 2$ , then  $\pi \in L_{\lambda+\varepsilon}^2(t, +\infty; V)$ ,  $\forall \varepsilon > 0$  follows promptly from (4.8) (4.9).  $\square$

**Theorem 4.3.** *If  $\pi : [t, +\infty) \rightarrow V$  satisfies (4.1) almost everywhere in  $[t, +\infty)$  and (4.2) then  $\pi$  is also a mild solution to (4.1)(4.2).*

*Proof.* By variation of constants formula, any  $\pi$  satisfying (4.1) a.e. is given by

$$(4.10) \quad \pi(\tau) = e^{(A_0^* - \lambda)(T-\tau)} \pi(T) + \int_{\tau}^T e^{(A_0^* - \lambda)(\sigma-\tau)} g'_0(y(\sigma)) d\sigma, \quad \forall T > t, \quad \forall \tau \in [t, T].$$

Note that (4.2) implies

$$\lim_{T \rightarrow +\infty} |e^{(A_0^* - \lambda)(T-\tau)} \pi(T)|_V \leq \lim_{T \rightarrow +\infty} e^{(\omega - \lambda)(T-\tau)} |\pi(T)|_V = 0.$$

hence by passing to limits as  $T \rightarrow +\infty$  in (4.10) one derives

$$\pi(\tau) = \lim_{T \rightarrow +\infty} \int_{\tau}^T e^{(A_0^* - \lambda)(\sigma-\tau)} g'_0(y(\sigma)) d\sigma = \int_{\tau}^{+\infty} e^{(A_0^* - \lambda)(\sigma-\tau)} g'_0(y(\sigma)) d\sigma$$

where the last equality follows from estimates (4.5)(4.6).  $\square$

**Remark 4.4.** Assume  $p \geq 2$ ,  $\lambda > 2\omega$ . By means of the same estimates contained in the proof of Lemma 4.2, one may show that

- $\lambda \leq \omega p$  implies  $y \in L^r(t, +\infty; V')$  for all  $r < \frac{\lambda}{p}$ , and  $y \in L^{\frac{\lambda}{p}}(t, T; V')$  for all  $T < +\infty$ ;
- $\lambda > \omega p$  implies  $y \in L^r(t, +\infty; V')$  for all  $r < p$ , and  $y \in L^p(t, T; V')$  for all  $T < +\infty$ .

**Definition 4.5.** Let Assumption 2.1 [1-7] be satisfied, and assume  $p \geq 2$  and  $\lambda > 2\omega$ . Let also  $T > t$  be either finite or  $+\infty$ . We say that a given pair  $(u, y) \in L_\lambda^p(t, T; U) \times L_{loc}^1(t, T; V')$  is extremal if and only if there exists a function  $\pi \in L_\lambda^q(t, T; V)$  satisfying in mild sense, along with  $u$  and  $y$ , the following set of equations

$$\begin{aligned} y'(\tau) &= Ay(\tau) + Bu(\tau), \quad \tau \in [t, T]; \quad y(t) = x \\ \pi'(\tau) &= (\lambda - A_0^*)\pi(\tau) - g_0'(y(\tau)), \quad \tau \in [t, T]; \\ \lim_{s \rightarrow +\infty} \pi(s) &= 0, \text{ when } T = +\infty; \quad \pi(T) = 0, \text{ when } T < +\infty \\ (4.11) \quad -B^*\pi(\tau) &\in \partial h_0(u(\tau)), \text{ for a.a. } \tau \in [t, T]. \end{aligned}$$

**Remark 4.6.** Note that, by conjugation formula, we have

$$-B^*\pi(\tau) \in \partial h_0(u(\tau)) \iff u(\tau) = (h_0^*)'(-B^*\pi(\tau)).$$

We refer to such condition as to maximum principle condition.

The next goal is to characterize optimal couples as extremal couples for the system. Then the following theorem holds true.

**Theorem 4.7. (Maximum Principle).** Let Assumptions 2.1 [1-7] be satisfied,  $\lambda > 2\omega$ . Then, for all  $p \geq 2$  and  $T < +\infty$ , the couple  $(u^*, y^*)$  is optimal at  $(t, x)$  - for the problem of minimizing (1.1)(2.2) - if and only if it is extremal.

*Proof.* Let  $K : L_\lambda^p(t, T; U) \rightarrow \mathbb{R} \cup \{+\infty\}$ , and  $G : L_\lambda^p(t, T; U) \rightarrow \mathbb{R} \cup \{+\infty\}$  be defined by

$$K(u) \equiv \int_t^T e^{-\lambda\tau} h_0(u(\tau)) d\tau, \quad G(u) \equiv \int_t^T e^{-\lambda\tau} g_0(y(\tau; t, x, u)) d\tau$$

so that  $\forall u \in \text{dom}(K) \cap \text{dom}(G)$  we have

$$J(u) = K(u) + G(u), \text{ and } \partial J(u) = \partial K(u) + \partial G(u).$$

*Claim 1:*

$$(4.12) \quad \begin{aligned} \partial K(u) &= S, \text{ where} \\ S &\equiv \{\varphi \in L_\lambda^q(t, T; U) : \varphi(\tau) \in \partial h_0(u(\tau)), \text{ a.a. } \tau \in [t, T]\} \end{aligned}$$

Indeed

$$(4.13) \quad \begin{aligned} \partial K(u) &= \left\{ \varphi \in L_\lambda^q(t, T; U) : \right. \\ &\quad \left. \int_t^T [h_0(w(\tau)) - h_0(u(\tau)) - (\varphi(\tau)|w(\tau) - u(\tau))_U] d\tau \geq 0, \quad \forall w \in L_\lambda^p(t, T; U) \right\}, \end{aligned}$$

so that  $S \subset \partial K(u)$  is straightforward. To show the reverse inclusion, we let  $v$  be any fixed element of  $\partial k(u)$ ,  $E$  any measurable subset of  $[t, T]$ , and we set

$$(4.14) \quad \tilde{w}(\tau) = \begin{cases} u(\tau), & \tau \notin E \\ w(\tau), & \tau \in E \end{cases}$$

We then derive

$$\int_E [h_0(w(\tau)) - h_0(u(\tau)) - (\varphi(\tau)|w(\tau) - u(\tau))_U] d\tau \geq 0, \quad \forall w \in L_\lambda^p(t, T; U),$$

which implies, since  $E$  and  $w$  where arbitrarily chosen,

$$h_0(w(\tau)) - h_0(u(\tau)) - (\varphi(\tau)|w(\tau) - u(\tau))_U \geq 0, \quad a.a. \tau \in [0, +\infty)$$

that is,  $\varphi(\tau) \in \partial h_0(\tau)$  for almost all  $\tau$ . On the other hand:

*Claim 2: Assume that  $(u, y)$  is extremal. Then  $(u, y)$  is optimal.*

Let  $p$  be the co-state associated to the extremal couple  $(u, y)$ , and let  $v$  be any control in  $\text{dom}(G)$ . Then

$$\begin{aligned} G(v) - G(u) &= \int_t^T [g_0(y(\tau; v)) - g_0(y(\tau; u))] e^{-\lambda\tau} d\tau \\ &\geq \int_t^T \langle g'_0(y(\tau; u)), \int_t^\tau e^{(\tau-\sigma)A} B(v(\sigma) - u(\sigma)) d\sigma \rangle_{V, V'} e^{-\lambda\tau} d\tau \\ &= \int_t^T \left( \int_\sigma^T B^* e^{(\tau-\sigma)A_0^*} g'_0(y(\tau; u)) |v(\sigma) - u(\sigma)\rangle_U e^{-\lambda\tau} d\tau d\sigma \right) \\ &= \int_t^T \left( \int_\sigma^T B^* e^{(\tau-\sigma)(A_0^* - \lambda)} g'_0(y(\tau; u)) |v(\sigma) - u(\sigma)\rangle_U e^{-\lambda\sigma} d\sigma \right) \\ &= \langle B^* \pi, v \rangle_{L_\lambda^q, L_\lambda^p} \end{aligned}$$

where we could exchange the order of integration due to the fact that  $\pi$  associated to an extremal couple is in  $L_\lambda^q(t, +\infty; V)$  by definition. Then we proved that

$$B^* \pi \in \partial G(u).$$

Since by assumption we also know that  $-B^* \pi(\sigma) \in \partial h_0(u(\sigma))$  almost everywhere in  $[t, +\infty[$ , we have also  $-B^* \pi \in \partial K(u)$  and hence  $\partial J(u) \ni 0$ , that is  $u$  is optimal.

*Claim 3: Assume that  $(u^*, y^*)$  is optimal, and let  $\pi^*$  be the associated costate. Then  $G$  is Gâteaux differentiable in  $u^*$  with  $G'(u^*) = B^* \pi^*$ .*

Indeed, for any fixed  $v$  in  $L_\lambda^p(t, +\infty; U)$ , and any  $\epsilon > 0$ , there exists  $0 < \epsilon_0 \leq \epsilon$  such that

$$\begin{aligned}
& \left| \frac{G(u^* + \epsilon v) - G(u^*)}{\epsilon} - \langle B^* \pi^*, v \rangle_{L_\lambda^q, L_\lambda^p} \right| = \\
& = \left| \int_t^T \frac{g_0(y(\tau; u^* + \epsilon v)) - g_0(y^*(\tau))}{\epsilon} e^{-\lambda \tau} d\tau - \langle B^* \pi^*, v \rangle_{L_\lambda^q, L_\lambda^p} \right| = \\
& = \left| \int_t^T \langle g'_0(y(\tau; u^* + \epsilon_0 v)) - g'_0(y^*(\tau)), \int_t^\tau e^{(\tau-\sigma)A} B v(\sigma) d\sigma \rangle_{V, V'} e^{-\lambda \tau} d\tau \right| \\
& \leq [g'_0] \epsilon_0 \int_t^T \left| \int_t^\tau e^{(\tau-\sigma)A} B v(\sigma) d\sigma \right|_{V'}^2 e^{-\lambda \tau} d\tau =: I
\end{aligned}$$

We estimate now the right hand side in the case  $\lambda \neq \omega p$ . By Hölder inequality one has

$$\begin{aligned}
\left| \int_t^\tau e^{(\tau-\sigma)A} B v(\sigma) d\sigma \right|_{V'} & \leq e^{\omega \tau} \left[ \int_t^\tau e^{(\frac{\lambda}{p} - \omega) q \sigma} d\sigma \right]^{\frac{1}{q}} \|B\|_{L(U, V')} \|v\|_{L_\lambda^p(t, \tau; U)} \\
& = e^{\omega \tau} \left| \frac{e^{(\frac{\lambda}{p} - \omega) q \tau} - e^{(\frac{\lambda}{p} - \omega) q t}}{q \left( \frac{\lambda}{p} - \omega \right)} \right|^{\frac{1}{q}} \|B\|_{L(U, V')} \|v\|_{L_\lambda^p(t, +\infty; U)}
\end{aligned}$$

Then, for a suitable constant  $C_3$  we have

$$\begin{aligned}
(4.15) \quad I & \leq \frac{[g'_0] \epsilon_0 \|B\|_{L(U, V')}^2 \|v\|_{L_\lambda^p(t, +\infty; U)}^2}{\left| \frac{\lambda}{p} - \omega \right|^{\frac{2}{q}} q^{\frac{2}{q}}} \int_t^\tau \left| e^{(\frac{\lambda}{p} - \omega) q \tau} - e^{(\frac{\lambda}{p} - \omega) q t} \right|^{\frac{2}{q}} e^{(\omega - \lambda) \tau} d\tau \\
& \leq C_3 \epsilon_0 \int_t^\tau \left( e^{(\frac{\lambda}{p} - \omega) q \tau} \vee 1 \right)^{\frac{2}{q}} e^{(\omega - \lambda) \tau} d\tau \\
& = C_3 \epsilon_0 \int_t^\tau e^{[\lambda(\frac{2}{p} - 1) - \omega] \tau} \vee e^{(\omega - \lambda) \tau} d\tau,
\end{aligned}$$

and since  $p \geq 2 \Rightarrow \lambda \left( \frac{2}{p} - 1 \right) - \omega < 0$ , then one may let  $\epsilon \rightarrow 0$  and obtains that the right hand side in (4.15) goes to 0. If instead  $\lambda = \omega p$  by Hölder inequality one has

$$\begin{aligned}
\left| \int_t^\tau e^{(\tau-\sigma)A} B v(\sigma) d\sigma \right|_{V'} & \leq e^{\omega \tau} \int_t^\tau e^{-\frac{\lambda}{p} \sigma} |v(\sigma)|_U d\sigma \|B\|_{L(U, V')} \|v\|_{L_\lambda^p(t, \tau; U)} \\
& \leq e^{\omega \tau} |\tau - t|^{\frac{1}{q}} \|B\|_{L(U, V')} \|v\|_{L_\lambda^p(t, T; U)}
\end{aligned}$$

Then

$$I \leq [g'_0] \epsilon_0 \|B\|_{L(U, V')}^2 \|v\|_{L_\lambda^p(t, T; U)}^2 \int_t^T |\tau - t|^{\frac{2}{q}} e^{-(\lambda - 2\omega) \tau} d\tau$$

and by letting  $\epsilon \rightarrow 0$  the right hand side goes to 0.

The proof of necessity is then straightforward: from optimality of  $u^*$  we have  $J(u^*) \ni 0$ , then from Claim 1 and Claim 3 we derive condition (4.11).  $\square$

**Lemma 4.8.** *Let  $(u^*, y^*)$  be optimal at  $(0, x)$  and let  $\pi^*(\cdot; 0, x)$  be the associated co-state. Then*

$$\Psi'(x) = \pi^*(0; 0, x).$$

Consequently,

$$\Psi'(y^*(\tau)) = \pi^*(\tau; \tau, y^*(\tau)).$$

*Proof.* To derive the first assertion it is sufficient to prove that  $p^*(0; 0, x)$  is in  $\partial\Psi(x)$ . We recall that in Theorem 4.7 we showed that  $-B^*\pi^*(\tau) \in \partial h_0(u^*(\tau))$  almost everywhere in  $[0, +\infty)$ . Then, for all  $\bar{x} \in V'$ , and their associated controls  $\bar{u}$  optimal at  $(0, \bar{x})$ , we have

$$\begin{aligned} \Psi(\bar{x}) - \Psi(x) &= J_\infty(0, \bar{x}, \bar{u}) - J(0, x, u^*) \\ &\geq \int_0^{+\infty} \left[ \langle g'_0(y^*(\tau)), \bar{y}(\tau) - y^*(\tau) \rangle_{V, V'} - (B^*\pi^*(\tau; 0, x)) | \bar{u}(\tau) - u^*(\tau) \rangle_U \right] e^{-\lambda\tau} d\tau. \end{aligned}$$

Note that

$$\begin{aligned} \langle g'_0(y^*(\tau)), \bar{y}(\tau) - y^*(\tau) \rangle_{V, V'} &= \\ &= \langle g'_0(y^*(\tau)), e^{A\tau}(\bar{x} - x) + \int_0^\tau e^{A(\tau-\sigma)} B(\bar{u}(\sigma) - u^*(\sigma)) d\sigma \rangle_{V, V'} \\ &= \langle B^* e^{A_0^*(\tau-\sigma)} g'_0(y^*(\tau)), \bar{x} - x \rangle_{V, V'} + (B^*\pi^*(\tau; 0, x)) | \bar{u}(\tau) - u^*(\tau) \rangle_U \end{aligned}$$

hence

$$\begin{aligned} \Psi(\bar{x}) - \Psi(x) &\geq \langle \int_0^{+\infty} B^* e^{(A_0^* - \lambda)\tau} g'_0(y^*(\tau)) e^{-\lambda\tau} d\tau, \bar{x} - x \rangle_{V, V'} \\ &= \langle \pi^*(0; 0, x), \bar{x} - x \rangle_{V, V'}. \end{aligned}$$

The proof of the second statement is standard and we write it here for the sake of completeness. Note that  $\pi(0; 0, y^*(\tau))$  is the costate associated to any couple, optimal at  $(0, y^*(\tau))$ , say  $u_{0, y^*(\tau)}$  and  $y(\cdot; 0, y^*(\tau), u_{0, y^*(\tau)})$ . Observe also that the control defined by

$$u_{0, y^*(\tau)}(\sigma) \equiv u^*(\sigma + \tau)$$

is optimal at  $(0, y^*(\tau))$ , and consequently the associated trajectory satisfies

$$y(\sigma; 0, y^*(\tau), u_{0, y^*(\tau)}) = y(\sigma + \tau; \tau, y^*(\tau), u^*) = y^*(\sigma + \tau).$$

Then

$$\begin{aligned} \Psi'(y^*(\tau)) &= p(0; 0, y^*(\tau)) \\ &= \int_0^{+\infty} e^{(A_0^* - \lambda)\sigma} g'_0(y(\sigma; 0, y^*(\tau), u_{0, y^*(\tau)})) d\sigma \\ &= \int_0^{+\infty} e^{(A_0^* - \lambda)\sigma} g'_0(y^*(\sigma + \tau)) d\sigma \\ &= \int_\tau^{+\infty} e^{(A_0^* - \lambda)(r - \tau)} g'_0(y(r)) dr \\ &= \pi(\tau; \tau, y^*(\tau)) \end{aligned}$$

□

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